Laplace's Equation in Spherical Polar Coördinates

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I.

We start with the primitive definitions

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

and

$$z = r \cos\theta$$

and their inverses

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = cos^{-1}\frac{z}{r} = cos^{-1}\frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

and

$$\phi = tan^{-1}\frac{y}{x}$$

and attempt to write (using the chain rule)

$$\frac{\partial}{\partial x} = \left(\frac{\partial r}{\partial x}\right)_{y,z}\left(\frac{\partial}{\partial r}\right)_{\theta,\phi} + \left(\frac{\partial \theta}{\partial x}\right)_{y,z}\left(\frac{\partial}{\partial \theta}\right)_{r,\phi} + \left(\frac{\partial \phi}{\partial x}\right)_{y,z}\left(\frac{\partial}{\partial \phi}\right)_{r,\theta}$$

and

$$\frac{\partial}{\partial y} = \left(\frac{\partial r}{\partial y}\right)_{x,z}\left(\frac{\partial}{\partial r}\right)_{\theta,\phi} + \left(\frac{\partial \theta}{\partial y}\right)_{x,z}\left(\frac{\partial}{\partial \theta}\right)_{r,\phi} + \left(\frac{\partial \phi}{\partial y}\right)_{x,z}\left(\frac{\partial}{\partial \phi}\right)_{r,\theta}$$

and

$$\frac{\partial}{\partial z} = \left(\frac{\partial r}{\partial z}\right)_{x,y}\left(\frac{\partial}{\partial r}\right)_{\theta,\phi} + \left(\frac{\partial \theta}{\partial z}\right)_{x,y}\left(\frac{\partial}{\partial \theta}\right)_{r,\phi} + \left(\frac{\partial \phi}{\partial z}\right)_{x,y}\left(\frac{\partial}{\partial \phi}\right)_{r,\theta}$$

The needed (above) partial derivatives are:

$$\left(\frac{\partial r}{\partial x}\right)_{y,z} = \sin\theta\cos\phi \tag{1}$$

$$\left(\frac{\partial r}{\partial y}\right)_{x,z} = \sin\theta\sin\phi \tag{2}$$

$$\left(\frac{\partial r}{\partial z}\right)_{x,y} = \cos\theta \tag{3}$$

and we have as a starting point for doing the θ terms,

$$d\cos\theta = -\sin\theta d\theta = \frac{dz}{r} - \frac{z}{r^2}\frac{1}{r}\left(xdx + ydy + zdz\right)$$

so that, for example

$$-\sin\theta d\theta = -\frac{z}{r^2}\frac{x}{r}dx$$

which is

$$-\sin\theta d\theta = -\frac{r\cos\theta}{r^2}\sin\theta\cos\phi dx$$

so that

$$\left(\frac{\partial\theta}{\partial x}\right)_{y,z} = \frac{\cos\theta\cos\phi}{r} \tag{4}$$

$$\left(\frac{\partial\theta}{\partial y}\right)_{x,z} = \frac{\cos\theta\sin\phi}{r} \tag{5}$$

but, for the z-equation, we have

$$-\sin\theta d\theta = \frac{dz}{r} - \frac{z}{r^2}\frac{1}{r}zdz$$

which is

$$-\sin\theta d\theta = \left(\frac{1}{r} - \frac{z^2}{r^3}\right)dz = \frac{r^2 - z^2}{r^3}dz$$

$$-\sin\theta d\theta = \left(\frac{1}{r} - \frac{z^2}{r^3}\right)dz = \frac{r^2\sin^2\theta}{r^3}dz$$

so one has

$$\left(\frac{\partial\theta}{\partial z}\right)_{x,y} = -\frac{\sin\theta}{r} \tag{6}$$

Next, we have (as an example)

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \tan^{-1}\frac{y}{x}$$

so

$$\left(1 + \frac{\sin^2\phi}{\cos^2\phi}\right)d\phi = \frac{dy}{x} - \frac{y}{x^2}dx$$

or

$$\left(\frac{1}{\cos^2\phi}\right)d\phi = \frac{dy}{x} - \frac{y}{x^2}dx$$

which leads to

$$\left(\frac{\partial\phi}{\partial y}\right)_{x,z} = \frac{\cos\phi}{r\sin\theta} \tag{7}$$

and

$$\left(\frac{\partial\phi}{\partial x}\right)_{y,z} = -\frac{\sin\phi}{r\sin\theta} \tag{8}$$

$$\left(\frac{\partial\phi}{\partial z}\right)_{x,y} = 0 \tag{9}$$

Given these results (above) we write

$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \left(\frac{\sin\theta}{r}\right)\frac{\partial}{\partial\theta} \tag{10}$$

and

$$\frac{\partial}{\partial y} = (\sin\theta\sin\phi)\frac{\partial}{\partial r} + \left(\frac{\cos\theta\sin\phi}{r}\right)\frac{\partial}{\partial\theta} + \left(\frac{\cos\phi}{r\sin\theta}\right)\frac{\partial}{\partial\phi} \tag{11}$$

and

$$\frac{\partial}{\partial x} = (\sin\theta\cos\phi)\frac{\partial}{\partial r} + \left(\frac{\cos\theta\cos\phi}{r}\right)\frac{\partial}{\partial\theta} + \left(-\frac{\sin\phi}{r\sin\theta}\right)\frac{\partial}{\partial\phi} \tag{12}$$

From Equation 10 we form

$$\frac{\partial^2}{\partial z^2} = \cos\theta\frac{\partial\left[\cos\theta\frac{\partial}{\partial r} - \left(\frac{\sin\theta}{r}\right)\frac{\partial}{\partial\theta}\right]}{\partial r} - \left(\frac{\sin\theta}{r}\right)\frac{\partial\left(\cos\theta\frac{\partial}{\partial r} - \left(\frac{\sin\theta}{r}\right)\frac{\partial}{\partial\theta}\right]}{\partial\theta} \tag{13}$$

while from Equation 11 we obtain

$$\begin{aligned}\frac{\partial^2}{\partial y^2} = (\sin\theta\sin\phi)\,&\frac{\partial\left[\sin\theta\sin\phi\frac{\partial}{\partial r} + \left(\frac{\cos\theta\sin\phi}{r}\right)\frac{\partial}{\partial\theta} + \left(\frac{\cos\phi}{r\sin\theta}\right)\frac{\partial}{\partial\phi}\partial r\right]}{\partial r}\\ + \left(\frac{\cos\theta\sin\phi}{r}\right)&\frac{\partial\left[\sin\theta\sin\phi\frac{\partial}{\partial r} + \left(\frac{\cos\theta\sin\phi}{r}\right)\frac{\partial}{\partial\theta} + \left(\frac{\cos\phi}{r\sin\theta}\right)\frac{\partial}{\partial\phi}\right]}{\partial\theta}\\ + \left(\frac{\cos\phi}{r\sin\theta}\right)&\frac{\partial\left[\sin\theta\sin\phi\frac{\partial}{\partial r} + \left(\frac{\cos\theta\sin\phi}{r}\right)\frac{\partial}{\partial\theta} + \left(\frac{\cos\phi}{r\sin\theta}\right)\frac{\partial}{\partial\phi}\right]}{\partial\phi}\end{aligned} \tag{14}$$

and from Equation 12 we obtain

$$\begin{aligned}\frac{\partial^2}{\partial x^2} = (\sin\theta\cos\phi)\,&\frac{\partial\left[\sin\theta\cos\phi\frac{\partial}{\partial r} + \left(\frac{\cos\theta\cos\phi}{r}\right)\frac{\partial}{\partial\theta} - \left(\frac{\sin\phi}{r\sin\theta}\right)\frac{\partial}{\partial\phi}\right]}{\partial r}\\ + \left(\frac{\cos\theta\cos\phi}{r}\right)&\frac{\partial\left[\sin\theta\cos\phi\frac{\partial}{\partial r} + \left(\frac{\cos\theta\cos\phi}{r}\right)\frac{\partial}{\partial\theta} - \left(\frac{\sin\phi}{r\sin\theta}\right)\frac{\partial}{\partial\phi}\right]}{\partial\theta}\\ - \left(\frac{\sin\phi}{r\sin\theta}\right)&\frac{\partial\left[\sin\theta\cos\phi\frac{\partial}{\partial r} + \left(\frac{\cos\theta\cos\phi}{r}\right)\frac{\partial}{\partial\theta} - \left(\frac{\sin\phi}{r\sin\theta}\right)\frac{\partial}{\partial\phi}\right]}{\partial\phi}\end{aligned} \tag{15}$$

Expanding, we have

$$\begin{aligned}\frac{\partial^2}{\partial z^2} &= \cos^2\theta\frac{\partial^2}{\partial r^2} + \frac{\cos\theta\sin\theta}{r^2}\frac{\partial}{\partial\theta} - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta}\\ -\left(\frac{\sin\theta}{r}\right)&\left(-\sin\theta\frac{\partial}{\partial r} - \cos\theta\frac{\partial}{\partial\theta}\right) - \frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} + \left(\frac{\sin\theta}{r}\right)^2\frac{\partial^2}{\partial\theta^2}\end{aligned} \tag{16}$$

while for the y-equation we have

$$\frac{\partial^2}{\partial y^2} = \sin^2\theta\sin^2\phi\frac{\partial^2}{\partial r^2} \tag{17}$$

$$+\sin\theta\sin\phi\left[+\left(\frac{\cos\theta\sin\phi}{r^2}\right)\frac{\partial}{\partial\theta} + \left(\frac{\cos\theta\sin\phi}{r}\right)\frac{\partial^2}{\partial r\partial\theta}\right] \tag{18}$$

$$+\sin\theta\sin\phi\left[\left(-\frac{\cos\phi}{r^2\sin\theta}\right)\frac{\partial}{\partial\phi}+\left(\frac{\cos\phi}{r\sin\theta}\right)\frac{\partial^2}{\partial r\partial\phi}\right] \tag{19}$$

$$+\left(\frac{\cos\theta\sin\phi}{r}\right)\left[\cos\theta\sin\phi\frac{\partial}{\partial r}+\sin\theta\sin\phi\frac{\partial^2}{\partial r\partial\theta}\right] \tag{20}$$

$$+\left(\frac{\cos\theta\sin\phi}{r}\right)\left[-\left(\frac{\sin\theta\sin\phi}{r}\right)\frac{\partial}{\partial\theta}+\left(\frac{\cos\theta\sin\phi}{r}\right)\frac{\partial^2}{\partial\theta^2}\right] \tag{21}$$

$$+\left(\frac{\cos\theta\sin\phi}{r}\right)\left[-\left(\frac{\cos\phi\cos\theta}{r\sin^2\theta}\right)\frac{\partial}{\partial\phi}+\left(\frac{\cos\phi}{r\sin\theta}\right)\frac{\partial^2}{\partial\phi\partial\theta}\right] \tag{22}$$

$$+\left(\frac{\cos\phi}{r\sin\theta}\right)\left[\sin\theta\cos\phi\frac{\partial}{\partial r}+\sin\theta\sin\phi\frac{\partial^2}{\partial r\partial\phi}\right] \tag{23}$$

$$+\left(\frac{\cos\phi}{r\sin\theta}\right)\left[+\left(\frac{\cos\theta\cos\phi}{r}\right)\frac{\partial}{\partial\theta}+\left(\frac{\cos\theta\sin\phi}{r}\right)\frac{\partial^2}{\partial\theta\partial\phi}\right] \tag{24}$$

$$+\left(\frac{\cos\phi}{r\sin\theta}\right)\left[-\left(\frac{\sin\phi\cos\phi}{r\sin\theta}\right)\frac{\partial}{\partial\phi}+\left(\frac{\cos\phi}{r\sin\theta}\right)\frac{\partial^2}{\partial\phi^2}\right] \tag{25}$$

and finally

$$\frac{\partial^2}{\partial x^2}=(\sin\theta\cos\phi)\sin\theta\cos\phi\frac{\partial^2}{\partial r^2}$$

$$+(\sin\theta\cos\phi)\left[-\left(\frac{\cos\theta\cos\phi}{r^2}\right)\frac{\partial}{\partial\theta}+\left(\frac{\cos\theta\cos\phi}{r}\right)\frac{\partial^2}{\partial\theta\partial r}\right] \tag{26}$$

$$-(\sin\theta\cos\phi)\left[-\left(\frac{\sin\phi}{r^2\sin\theta}\right)\frac{\partial}{\partial\phi}+\left(\frac{\sin\phi}{r\sin\theta}\right)\frac{\partial^2}{\partial\phi\partial r}\right] \tag{27}$$

$$+\left(\frac{\cos\theta\cos\phi}{r}\right)\left[\cos\theta\cos\phi\frac{\partial}{\partial r}+\sin\theta\cos\phi\frac{\partial^2}{\partial r\partial\theta}\right] \tag{28}$$

$$+\left(\frac{\cos\theta\cos\phi}{r}\right)\left[-\left(\frac{\sin\theta\cos\phi}{r}\right)\frac{\partial}{\partial\theta}+\left(\frac{\cos\theta\cos\phi}{r}\right)\frac{\partial^2}{\partial\theta^2}\right] \tag{29}$$

$$+\left(\frac{\cos\theta\cos\phi}{r}\right)\left[+\left(\frac{\sin\phi}{r\sin^2\theta}\right)\frac{\partial}{\partial\phi}-\left(\frac{\sin\phi}{r\sin\theta}\right)\frac{\partial^2}{\partial\phi\partial\theta}\right] \tag{30}$$

$$-\left(\frac{\sin\phi}{r\sin\theta}\right)\left[\sin\theta\sin\phi\frac{\partial}{\partial r}+\sin\theta\cos\phi\frac{\partial^2}{\partial r\partial\phi}\right] \tag{31}$$

$$-\left(\frac{\sin\phi}{r\sin\theta}\right)\left[-\left(\frac{\cos\theta\sin\phi}{r}\right)\frac{\partial}{\partial\theta}+\left(\frac{\cos\theta\cos\phi}{r}\right)\frac{\partial^2}{\partial\theta\partial\phi}\right] \tag{32}$$

$$-\left(\frac{\sin\phi}{r\sin\theta}\right)\left[-\left(\frac{\cos\phi}{r\sin\theta}\right)\frac{\partial}{\partial\phi}-\left(\frac{\sin\phi}{r\sin\theta}\right)\frac{\partial^2}{\partial\phi^2}\right] \tag{33}$$

Now, one by one, we expand completely each of these three terms. We have

$$\frac{\partial^2}{\partial z^2}=\cos^2\theta\frac{\partial^2}{\partial r^2} \tag{34}$$

$$+\frac{\cos\theta\sin\theta}{r^2}\frac{\partial}{\partial\theta} \tag{35}$$

$$-\frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta} \tag{36}$$

$$+\left(\frac{\sin^2\theta}{r}\right)\frac{\partial}{\partial r} \tag{37}$$

$$-\left(\frac{\sin\theta\cos\theta}{r}\right)\frac{\partial^2}{\partial r\partial\theta} \tag{38}$$

$$+\frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} \tag{39}$$

$$+\left(\frac{\sin^2\theta}{r^2}\right)\frac{\partial^2}{\partial\theta^2} \tag{40}$$

and, for the y-equation:

$$\frac{\partial^2}{\partial y^2} = \sin^2\theta \sin^2\phi \frac{\partial^2}{\partial r^2} \tag{41}$$

$$(18) \to + \left(\frac{\sin\theta\cos\theta\sin^2\phi}{r^2}\right)\frac{\partial}{\partial\theta} \tag{42}$$

$$+ \left(\frac{\cos\theta\sin\theta\sin^2\phi}{r}\right)\frac{\partial^2}{\partial r\partial\theta} \tag{43}$$

$$(19) \to - \left(\frac{\sin\phi\cos\phi}{r^2}\right)\frac{\partial}{\partial\phi} \tag{44}$$

$$+ \left(\frac{\cos\phi\sin\phi}{r}\right)\frac{\partial^2}{\partial r\partial\phi} \tag{45}$$

$$(20) \to + \left(\frac{\cos^2\theta\sin^2\phi}{r}\right)\frac{\partial}{\partial r} \tag{46}$$

$$+ \left(\frac{\cos\theta\sin\theta\sin^2\phi}{r}\right)\frac{\partial^2}{\partial r\partial\theta} \tag{47}$$

$$- \left(\frac{\sin\theta\cos\theta\sin^2\phi}{r^2}\right)\frac{\partial}{\partial\theta} \tag{48}$$

$$(21) \to + \left(\frac{\cos^2\theta\sin^2\phi}{r^2}\right)\frac{\partial^2}{\partial\theta^2} \tag{49}$$

$$- \left(\frac{\cos^2\theta\cos\phi\sin\phi}{r\sin^2\theta}\right)\frac{\partial}{\partial\phi} \tag{50}$$

$$+ \left(\frac{\cos\theta\cos\phi\sin\phi}{r^2\sin\theta}\right)\frac{\partial^2}{\partial\phi\partial\theta} \tag{51}$$

$$(22) \to + \left(\frac{\cos^2\phi}{r}\right)\frac{\partial}{\partial r} \tag{52}$$

$$+ \left(\frac{\cos\phi\sin\phi}{r}\right)\frac{\partial^2}{\partial r\partial\phi} \tag{53}$$

$$+ \left(\frac{\cos^2\phi\cos\theta}{r^2\sin\theta}\right)\frac{\partial}{\partial\theta} \tag{54}$$

$$(24) \to + \left(\frac{\cos\theta\cos\phi\sin\phi}{r^2\sin\theta}\right)\frac{\partial^2}{\partial\theta\partial\phi} \tag{55}$$

$$(25) \to - \left(\frac{\cos^2\phi\sin\phi}{r\sin^2\theta}\right)\frac{\partial}{\partial\phi} \tag{56}$$

$$+ \left(\frac{\cos^2\phi}{r^2\sin^2\theta}\right)\frac{\partial^2}{\partial\phi^2} \tag{57}$$

and finally, for the x-equation, we have

$$\frac{\partial^2}{\partial x^2} = \sin^2\theta\cos^2\phi\frac{\partial^2}{\partial r^2} \tag{58}$$

$$(26) \to - \left(\frac{\sin\theta\cos\theta\cos^2\phi}{r^2}\right)\frac{\partial}{\partial\theta} \tag{59}$$

$$(26) \to + \left(\frac{\sin\theta\cos\theta\cos^2\phi}{r}\right)\frac{\partial^2}{\partial\theta\partial r} \tag{60}$$

$$\left(\frac{\cos\phi\sin\phi}{r^2}\right)\frac{\partial}{\partial\phi} \tag{61}$$

$$- \left(\frac{\sin\phi\cos\phi}{r}\right)\frac{\partial^2}{\partial\phi\partial r} \tag{62}$$

$$(27) \rightarrow +\left(\frac{\cos^2\theta\cos^2\phi}{r}\right)\frac{\partial}{\partial r} \tag{63}$$

$$+\left(\frac{\sin\theta\cos\theta\cos^2\phi}{r}\right)\frac{\partial^2}{\partial r\partial\theta} \tag{64}$$

$$(27) \rightarrow -\left(\frac{\sin\theta\cos\theta\cos^2\phi}{r^2}\right)\frac{\partial}{\partial\theta} \tag{65}$$

$$+\left(\frac{\cos^2\theta\cos^2\phi}{r^2}\right)\frac{\partial^2}{\partial\theta^2} \tag{66}$$

$$(28) \rightarrow +\left(\frac{\cos\theta\cos\phi}{r}\right)\left(\frac{\cos\phi\sin\phi}{r\sin\theta}\right)\frac{\partial}{\partial\phi} \tag{67}$$

$$-\left(\frac{\sin\phi\cos\phi\cos\theta}{r^2\sin\theta}\right)\frac{\partial^2}{\partial\phi\partial\theta} \tag{68}$$

$$(29) \rightarrow -\left(\frac{\sin^2\phi}{r}\right)\frac{\partial}{\partial r} \tag{69}$$

$$-\left(\frac{\sin\phi\cos\phi}{r}\right)\frac{\partial^2}{\partial r\partial\phi} \tag{70}$$

$$(31) \rightarrow +\left(\frac{\cos\theta\sin^2\phi}{r^2\sin\theta}\right)\frac{\partial}{\partial\theta} \tag{71}$$

$$-\left(\frac{\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\right)\frac{\partial^2}{\partial\theta\partial\phi} \tag{72}$$

$$(32) \rightarrow +\left(\frac{\sin\phi\cos\phi}{r\sin^2\theta}\right)\frac{\partial}{\partial\phi} \tag{73}$$

$$+\left(\frac{\sin^2\phi}{r^2\sin^2\theta}\right)\frac{\partial^2}{\partial\phi^2} \tag{74}$$

Gathering terms as coefficients of partial derivatives, we obtain (from Equations 34, 41 and 58)

$$\frac{\partial^2}{\partial r^2}\left(\cos^2\theta+\sin^2\theta\sin^2\phi+\sin^2\theta\cos^2\phi\right) \rightarrow \frac{\partial^2}{\partial r^2}$$

and (from Equations 35, 38, 42, 48, 54, 59, 65, and 71)

$$\frac{\partial}{\partial\theta}\Bigg(+\frac{\cos\theta\sin\theta}{r^2}+\frac{\sin\theta\cos\theta}{r^2}-\frac{\sin\theta\cos\theta\sin^2\phi}{r^2}-\frac{\sin\theta\cos\theta\sin^2\phi}{r^2}+\frac{\cos^2\phi\cos\theta}{r^2\sin\theta}$$
$$-\frac{\sin\theta\cos\theta\cos^2\phi}{r^2}-\frac{\sin\theta\cos\theta\cos^2\phi}{r^2}+\frac{\cos\theta\sin^2\phi}{r^2\sin\theta}\Bigg)$$
$$\rightarrow \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta} \tag{75}$$

while we obtain from Equations 40, 49, and 66:

$$\frac{\partial^2}{\partial\theta^2}\left(\frac{\sin^2\theta}{r^2}+\frac{\cos^2\theta\sin^2\phi}{r^2}+\frac{\cos^2\theta\cos^2\phi}{r^2}\right) \rightarrow \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \tag{76}$$

From Equations 37, 46, 52, 63, 69,

$$\frac{\partial}{\partial r}\left(+\frac{\sin^2\theta}{r}+\frac{\cos^2\theta\sin^2\phi}{r}+\frac{\cos^2\phi}{r}+\frac{\cos^2\theta\cos^2\phi}{r}-\frac{\sin^2\phi}{r}\right) \rightarrow \frac{2}{r}\frac{\partial}{\partial r} \tag{77}$$

From Equations 44, 50, 56, 61, 67 and 73 we obtain

$$\frac{\partial}{\partial\phi}\Bigg(-\frac{\sin\phi\cos\phi}{r^2}-\frac{\cos^2\theta\cos\phi\sin\phi}{r\sin^2\theta}-\frac{\cos^2\theta\cos\phi\sin\phi}{r\sin^2\theta}+\frac{\cos\phi\sin\phi}{r^2}+\left(\frac{\cos\theta\cos\phi}{r}\right)$$
$$+\left(\frac{\cos\theta\cos^2\phi\sin\phi}{r^2\sin\theta}\right)+\frac{\sin\phi\cos\phi}{r\sin^2\theta}\Bigg) \rightarrow 0 \tag{78}$$

From Equations 57 and 74 we obtain

$$\frac{\partial^2}{\partial\phi^2}\left(\frac{\cos^2\phi}{r^2\sin^2\theta}+\frac{\sin^2\phi}{r^2\sin^2\theta}\right)\to\left(\frac{1}{r^2\sin^2\theta}\right)\frac{\partial^2}{\partial\phi^2} \tag{79}$$

The mixed derivatives yield, first, from Equations 45, 53, 62, and 70 leading to

$$\frac{\partial^2}{\partial r\partial\phi}\left(\frac{\cos\phi\sin\phi}{r}+\frac{\cos\phi\sin\phi}{r}-\frac{\sin\phi\cos\phi}{r}-\frac{\sin\phi\cos\phi}{r}\right)\to 0 \tag{80}$$

From Equations 36, 39, 47, 43 64, 60

$$\begin{aligned}\frac{\partial^2}{\partial r\partial\theta}\Bigg(&-\frac{\sin\theta\cos\theta}{r}-\frac{\sin\theta\cos\theta}{r}+\frac{\cos\theta\sin\theta\sin^2\phi}{r}\\ &+\frac{\sin\theta\cos\theta\cos^2\phi}{r}+\frac{\cos\theta\sin\theta\sin^2\phi}{r}+\frac{\sin\theta\cos\theta\cos^2\phi}{r}\Bigg)\to 0\end{aligned} \tag{81}$$

From Equations 51 55 68 72

$$\begin{aligned}\frac{\partial^2}{\partial\phi\partial\theta}\Bigg(&\frac{\cos\theta\cos\phi\sin\phi}{r^2\sin\theta}+\frac{\cos\phi\sin\phi}{r^2\sin\theta}\\ &-\left(\frac{\sin\phi\cos\phi\cos\theta}{r^2\sin\theta}\right)-\left(\frac{\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\right)\Bigg)\to 0\end{aligned} \tag{82}$$

Gathering together the non-vanishing terms, we obtain

$$\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

which is one of the two "classic" forms for ∇^2. The other is

$$\frac{1}{r^2}\frac{\partial\left(r^2\frac{\partial}{\partial r}\right)}{\partial r}+\frac{1}{r^2\sin^2\theta}\left(\sin\theta\frac{\partial\left(\sin\theta\frac{\partial}{\partial\theta}\right)}{\partial\theta}+\frac{\partial^2}{\partial\phi^2}\right)$$

II. MAPLE EQUIVALENT

Here is a set of Maple instructions which will get you the same result:

```
restart;
f:=g(r,theta,phi);
tx :=
sin(theta)*cos(phi)*diff(f,r)+((cos(theta)*cos(phi))/r)*diff(f,theta)
-(sin(phi)/(r*sin(theta)))*diff(f,phi);
tx2:=expand(
sin(theta)*cos(phi)*diff(tx,r)+((cos(theta)*cos(phi))/r)*diff(tx,theta)
-(sin(phi)/(r*sin(theta)))*diff(tx,phi));
ty :=
sin(theta)*sin(phi)*diff(f,r)+((cos(theta)*sin(phi))/r)*diff(f,theta)
+(cos(phi)/(r*sin(theta)))*diff(f,phi);

ty2:=expand(sin(theta)*sin(phi)*diff(ty,r)+((cos(theta)*sin(phi))/r)
*diff(ty,theta)+(cos(phi)/(r*sin(theta)))*diff(ty,phi));
tz := cos(theta)*diff(f,r)
-(sin(theta)/r)*diff(f,theta);
tz2 := expand(cos(theta)*diff(tz,r)-(sin(theta)/r)*diff(tz,theta));

del := tx2+ty2+tz2:
del := algsubs( cos(theta)^2=1-sin(theta)^2, del ):
del := expand(algsubs( cos(phi)^2=1-sin(phi)^2, del ));
```